Why Do We Invert and Multiply?

To multiply two fractions, we multiply the numerators to get the new numerator and multiply the denominators to get the new denominator. However, we are taught that when faced with a problem such as $3/5 \div 4/7$, we should invert the second fraction and multiply. It would certainly seem more intuitive simply to divide the numerator of the first fraction by the numerator of the second and then similarly to divide one denominator by the other. Would this intuitive method of solving the problem work? Not only is the answer "yes," using this method of dividing fractions actually sheds some light on why we are taught to invert and multiply! Let's try $3/5 \div 4/7$ as an example.

$$3/5 \div 4/7 = \frac{3 \div 4}{5 \div 7} = \frac{3/4}{5/7}$$

Since 1 is the identity element for multiplication, we can multiply our answer by $4/4$, which is equivalent to 1, in order to get a whole number for our numerator.

$$\frac{3/4}{5/7} \times \frac{4}{4} = \frac{3}{20/7}$$

Since $7/7$ is also equivalent to 1, we can multiply our answer by $7/7$ in order to get a whole number for our denominator.

$$\frac{3}{20/7} \times \frac{7}{7} = \frac{21}{20}$$

Believe it or not, this is the same answer we arrive at by inverting and multiplying. However, the intuitive method requires a lot more work to achieve the same results. Let's review what that method involved.

$$\underset{\text{Dividend}}{3/5} \div \underset{\text{Divisor}}{4/7} = \frac{3 \div 4\,(4 \times 7)}{5 \div 7\,(4 \times 7)} = \frac{21}{20}$$

After dividing 3 (the dividend's numerator) by 4, we multiplied by 4 and then 7. Since multiplying by 4 cancels division by 4, we may as well simply multiply by 7 (the divisor's denominator). Likewise, after dividing 5 (the dividend's denominator) by 7, we multiplied by 4 and then 7. Since multiplying by 7 cancels division by 7, we may as well simply multiply by 4 (the divisor's numerator).

$$3/5 \div 4/7 = \frac{3}{5} \times \frac{7}{4} = \frac{21}{20}$$

So, inverting and multiplying when dividing fractions is actually just a shortcut! Be sure to let your students know this; kids love shortcuts.

Alternate Algorithms for Dividing Fractions

Now that we know inverting and multiplying is simply a shortcut when dividing fractions, we can explore further and discover some interesting and useful information. For example, some problems (such as $5/12 \div 1/3$) can be solved more quickly and easily by simply dividing, as you can see below:

Using Division	Invert and Multiply
$5/12 \div 1/3 = \boxed{\frac{5 \div 1}{12 \div 3}} = 5/4$ This step may be completed mentally	$5/12 \div 1/3 = 5/12 \times 3/1 =$ $15/12 = \boxed{\frac{15 \div 3}{12 \div 3}} = 5/4$ This step may be completed mentally

Of even greater interest is the fact that an efficient algorithm using division can always be used for fractions that have a common denominator! Since any number divided by itself is one, you can simply discard the denominators and use the quotient of the numerators as your answer! (See the example below.)

$$9/17 \div 5/17 = \boxed{\frac{9 \div 5}{17 \div 17} = \frac{9/5}{1}} = 9/5$$

These steps may be completed mentally

The algorithm above can always be used to divide fractions if a common denominator is found first. The fact that this method requires finding common denominators (a skill vital in adding and subtracting fractions with unlike denominators) is an added bonus. Here is an example:

$$9/12 \div 5/8 = 18/24 \div 15/24 = \frac{18 \div 15}{1} = \frac{18}{15}\left(\div \frac{3}{3}\right) = 6/5$$

Many important concepts are involved in this exploration of the division of fractions. If you share this material with your students, they may benefit from sidebar lessons involving fractions as division (i.e.: $9 \div 5 = 9/5$), inverse operations (i.e.: multiplication by 4 cancels division by 4), and (fractions with a denominator of 1 being equivalent to the numerator (i.e.: $3/1 = 3$, and $\frac{9/5}{1} = 9/5$).